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Su(2) intelligent states as coupled *su*(2) coherent states

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Abstract
We show how *su*(2) intelligent states can be obtained by coupling *su*(2) coherent states. The construction is simple and efficient, and easily leads to a discussion of some general properties of *su*(2) intelligent states.

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1. Introduction

In quantum mechanics, uncertainty relations give a lower bound on the uncertainty resulting from the simultaneous measurement of two non-commuting observables. One common uncertainty relation was obtained in [1]: if $\hat{\Omega}$ and $\hat{\Lambda}$ are self-adjoint operators, and if $|\psi\rangle$ is a state normalized to 1, then we have

$$\Delta\Omega\Delta\Lambda \geqslant \tfrac{1}{2}|\langle[\hat{\Omega}, \hat{\Lambda}]\rangle|. \tag{1}$$

In equation (1), $\Delta\Omega$ is the standard deviation of the operator $\hat{\Omega}$ for a quantum system described by $|\psi\rangle$, i.e.

$$\Delta\Omega = \sqrt{\langle\hat{\Omega}^2\rangle - \langle\hat{\Omega}\rangle^2}, \tag{2}$$

with $\langle\hat{X}\rangle = \langle\psi|\hat{X}|\psi\rangle$.

In this paper, we will discuss *su*(2) states for which the strict equality in equation (1) holds, i.e. *su*(2) states for which ($\hbar = 1$)

$$\Delta L_x \Delta L_y = \tfrac{1}{2}|\langle\hat{L}_z\rangle|. \tag{3}$$

States that satisfy equation (3) are known as *su*(2) intelligent states. The terminology was first introduced by Aragone *et al* [2]. It is clear that the right-hand side of equation (3) depends on the choice of state used to evaluate $\langle\hat{L}_z\rangle$, so intelligent states need to be distinguished from minimum uncertainty states; there are intelligent states for which the rhs of equation (3) is not the obvious minimum value of 0.

By *su*(2) state, we understand a (pure) quantum state $|\psi\rangle$ that belongs to an irreducible representation of the *su*(2) algebra. This algebra is spanned by the familiar angular momentum

operators $\{\hat{L}_x, \hat{L}_y, \hat{L}_z\}$ or, more conveniently, by the complex linear combinations $\{\hat{L}_\pm, \hat{L}_z\}$, where

$$\hat{L}_\pm = \hat{L}_x \pm \mathrm{i}\hat{L}_y, \qquad [\hat{L}_z, \hat{L}_\pm] = \pm\hat{L}_\pm, \qquad [\hat{L}_+, \hat{L}_-] = 2\hat{L}_z. \tag{4}$$

An irreducible representation of dimension $2j+1$, where j can be an integer or a half-integer, is spanned by the set $\{|jm\rangle, m = -j, -j+1, \ldots, j-1, j\}$ with

$$\hat{L}_z|jm\rangle = m|jm\rangle, \qquad \hat{L}_\pm|jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|j, m \pm 1\rangle. \tag{5}$$

Intelligence is not limited to $su(2)$ states. A well-known example of intelligent states is the harmonic oscillator coherent state $|\xi\rangle$, parameterized by the complex number ξ and for which

$$\Delta x \Delta p = \tfrac{1}{2}. \tag{6}$$

However, in this paper, we understand intelligent states to mean $su(2)$ intelligent states.

The terminology '$su(2)$ intelligent states' is to be contrasted with recent theoretical and experimental work [3–5] on angular momentum states of light as quantum states carrying orbital angular momentum about the beam axis. In these papers, the spectrum of the operator $\hat{L}_z$ is unbounded, leading to a differential eigenvalue equation rather than the finite-dimensional eigenvalue problem of equation (12).

An important ingredient to our construction will be the $su(2)$ coherent states [6]. It is sufficient here to recall the well-known property that such states are obtained by a rotation of the extremal $su(2)$ state $|\ell, \ell\rangle$. More specifically, an $su(2)$ coherent state $|\gamma, \beta\rangle$ can be parameterized by two angles γ, β such that, up to an overall phase

$$|\gamma, \beta\rangle = R_z(\gamma)R_y(\beta)R_z(-\gamma)|\ell, \ell\rangle, \tag{7}$$

where $R_i(\varphi)$ denotes the rotation about the axis i by an angle φ. $Su(2)$ coherent states with $\gamma = 0$ or $\pi/2$ also satisfy equation (3). However, $su(2)$ intelligent states are not always of the form of equation (7).

Indeed, we plan to show that all $su(2)$ intelligent states are of the form

$$[R_y(\beta)|\ell_\mathrm{A}, \ell_\mathrm{A}\rangle] \otimes [R_y(-\beta)|\ell_\mathrm{B}, \ell_\mathrm{B}\rangle], \tag{8}$$

$$\text{or} \quad [R_x(\beta)|\ell_\mathrm{A}, \ell_\mathrm{A}\rangle] \otimes [R_x(-\beta)|\ell_\mathrm{B}, \ell_\mathrm{B}\rangle], \tag{9}$$

corresponding to equation (7) with $\gamma = 0$ or $\pi/2$ and a specific choice of β.

$Su(2)$ intelligent states of angular momentum ℓ are of the form

$$\begin{aligned} &\hat{\Pi}^\ell[R_y(\beta)|\ell_\mathrm{A}, \ell_\mathrm{A}\rangle] \otimes [R_y(-\beta)|\ell_\mathrm{B}, \ell_\mathrm{B}\rangle], \\ &\text{or} \quad \hat{\Pi}^\ell[R_x(\beta)|\ell_\mathrm{A}, \ell_\mathrm{A}\rangle] \otimes [R_x(-\beta)|\ell_\mathrm{B}, \ell_\mathrm{B}\rangle], \end{aligned} \tag{10}$$

with $\ell = \ell_\mathrm{A} + \ell_\mathrm{B}$ and where $\hat{\Pi}^\ell = \sum_m |\ell, m\rangle\langle\ell, m|$ is the (non-unitary) operator that projects into the ℓ subspace.

Thus, our work functions as a bridge between the work of Hillery and Mlodinow [7] and the work of Rashid [8]. In [7], some intelligent states were obtained as $su(2)$ coherent states. They correspond to setting $\ell_\mathrm{B} = 0$ in equation (10). No projection is required and, although not every $su(2)$ intelligent state can be constructed, the use of a single unitary transformation means that these states are amenable to experimental implementation [9]. The construction method of [8] is distinctive in that it requires the use of a non-unitary transformation, although it completely solves the construction problem in a single shot.

Equation (10), on the other hand, lends itself to a clear physical interpretation: to construct a general intelligent state of angular momentum ℓ, we must bring together two separate systems,

each of which has been subjected to a different unitary transformation, and then extract from this combined system states of good angular momentum using a non-unitary operation akin to a measurement of ℓ. This interpretation provides a much clearer picture of *su*(2) intelligent states than that presented in [10].

In addition to [7] and [8], the original work [2] of Aragone *et al* has blossomed in various directions. In particular, the recent work of [11] deals with entanglement and *su*(2) intelligent states. Generalized intelligent states, which satisfy

$$\Delta\Omega^2\Delta\Lambda^2 = \tfrac{1}{4}\langle[\hat{\Omega}, \hat{\Lambda}]\rangle^2 + \tfrac{1}{4}\langle\{\hat{\Omega} - \langle\Omega\rangle, \hat{\Lambda} - \langle\Lambda\rangle\}\rangle^2, \tag{11}$$

where $\{\hat{\Omega}, \hat{\Lambda}\} \equiv \hat{\Omega}\hat{\Lambda} + \hat{\Lambda}\hat{\Omega}$, have been the object of considerable attention (see, for instance, [12]), including various applications in quantum optics [13–15]. Several authors, in particular [16], have studied spin squeezing using the construction of [8]. Trifonov [17] has studied multi-observables and multidimensional generalizations of equation (11).

Our work is organized as follows. We first identify a simple but basic property of solutions of the eigenvalue problem; this is encapsulated in equation (24). Once this is done, the eigenvalue problem associated with intelligence is solved explicitly for spin-$\frac{1}{2}$ in section 3. These spin-$\frac{1}{2}$ states and equation (24) are used in section 4.1 to construct, using a minimum amount of extra work, all intelligent states of angular momentum $\ell = 5/2$. This method is generalized to arbitrary ℓ in section 4. The general expression for our angular momentum state can be found in equation (60). Some simple analytical and numerical results are presented in section 5. A discussion and a short conclusion can be found in section 6.

2. Some simple properties

Recall [18] that intelligent states $|\psi^\ell(\alpha)\rangle$ of angular momentum ℓ are eigenstates of the non-Hermitian operator $\hat{L}_x - \mathrm{i}\alpha\hat{L}_y$, i.e. they satisfy

$$(\hat{L}_x - \mathrm{i}\alpha\hat{L}_y)|\psi^\ell(\alpha)\rangle = \lambda|\psi^\ell(\alpha)\rangle, \tag{12}$$

where $-\infty \leqslant \alpha \leqslant \infty$ is a real parameter. The eigenvalue λ is related to the average value of $\hat{L}_x$ and $\hat{L}_y$ and to the parameter α via:

$$\lambda = \langle\hat{L}_x\rangle - \mathrm{i}\alpha\langle\hat{L}_y\rangle. \tag{13}$$

One reaches equation (12) in two steps. First, in order to replace the inequality of equation (1) by the equality and obtain equation (11), the states $\hat{L}_x - \langle\hat{L}_x\rangle)|\psi^\ell(\alpha)\rangle$ and $(\hat{L}_y - \langle\hat{L}_y\rangle)|\psi^\ell(\alpha)\rangle$ must be collinear, i.e.

$$(\hat{L}_x - \langle\hat{L}_x\rangle)|\psi^\ell(\alpha)\rangle = \mathrm{i}\alpha(\hat{L}_y - \langle\hat{L}_y\rangle)|\psi^\ell(\alpha)\rangle. \tag{14}$$

We obtain intelligence by forcing the anticommutator term in equation (11) to 0:

$$\langle\psi^\ell(\alpha)|\{\hat{L}_x - \langle\hat{L}_x\rangle, \hat{L}_y - \langle\hat{L}_y\rangle\}|\psi^\ell(\alpha)\rangle = 0. \tag{15}$$

This restricts the values of α to be real and produces equation (12).

Let us now abstractly consider a composite system made from two independent subsystems, denoted by the subscripts A and B respectively, such that

$$\hat{L}_{x,\mathrm{A}} \equiv \hat{L}_x \otimes \mathbb{1}_\mathrm{B}, \qquad \hat{L}_{x,\mathrm{B}} \equiv \mathbb{1}_\mathrm{A} \otimes \hat{L}_x, \tag{16}$$

$$\hat{L}_x = \hat{L}_{x,\mathrm{A}} + \hat{L}_{x,\mathrm{B}}, \tag{17}$$

where $\mathbb{1}_\mathrm{A}$ and $\mathbb{1}_\mathrm{B}$ are unit operators in their respective subspaces. Equation (16) simply means that $\hat{L}_{x,\mathrm{A}}$ acts on the first (or 'A') subsystem only, leaving the second (or 'B') subsystem alone,

and similarly for $\hat{L}_{x,\mathrm{B}}$. The operators

$$\hat{L}_y = \hat{L}_{y,\mathrm{A}} + \hat{L}_{y,\mathrm{B}}, \tag{18}$$

$$\hat{L}_z = \hat{L}_{z,\mathrm{A}} + \hat{L}_{z,\mathrm{B}} \tag{19}$$

are defined in a similar obvious manner.

Let $|\chi(\alpha)\rangle_\mathrm{A}$ and $|\phi(\alpha)\rangle_\mathrm{B}$ be states of subsystems A and B, respectively, with the property that

$$(\hat{L}_{x,\mathrm{A}} - \mathrm{i}\alpha\hat{L}_{y,\mathrm{A}})|\chi(\alpha)\rangle_\mathrm{A} = \lambda_\mathrm{A}|\chi(\alpha)\rangle_\mathrm{A} \tag{20}$$

$$(\hat{L}_{x,\mathrm{B}} - \mathrm{i}\alpha\hat{L}_{y,\mathrm{B}})|\phi(\alpha)\rangle_\mathrm{B} = \nu_\mathrm{B}|\phi(\alpha)\rangle_\mathrm{B}, \tag{21}$$

i.e. $|\chi\,(\alpha)\rangle_\mathrm{A}$ and $|\phi\,(\alpha)\rangle_\mathrm{B}$ are intelligent in their respective subsystems. Then,

$$|\psi(\alpha)\rangle = |\chi\,(\alpha)\rangle_\mathrm{A} \otimes |\phi\,(\alpha)\rangle_\mathrm{B} \equiv |\chi\,(\alpha)\rangle_\mathrm{A}\,|\phi\,(\alpha)\rangle_\mathrm{B} \tag{22}$$

is intelligent since

$$\begin{aligned}(\hat{L}_x - \mathrm{i}\alpha\hat{L}_y)|\psi(\alpha)\rangle &= [(\hat{L}_{x,\mathrm{A}} - \mathrm{i}\alpha\hat{L}_{y,\mathrm{A}})|\chi(\alpha)\rangle_\mathrm{A}]|\phi(\alpha)\rangle_\mathrm{B} \\ &\quad + |\chi(\alpha)\rangle_\mathrm{A}[(\hat{L}_{x,\mathrm{B}} - \mathrm{i}\alpha\hat{L}_{y,\mathrm{B}})|\phi(\alpha)\rangle_\mathrm{B}], \qquad (23)\\ &= (\lambda_\mathrm{A} + \nu_\mathrm{B})|\chi(\alpha)\rangle_\mathrm{A}|\phi(\alpha)\rangle_\mathrm{B}. \qquad (24)\end{aligned}$$

In other words, the direct product of two intelligent states is also intelligent, provided that one thinks of the resulting state as a composite state constructed from two separate systems. This simple result is quite powerful as it indicates that intelligent states can be 'built-up' by putting together other intelligent states.

Quite clearly, the task now at hand is to find the simplest intelligent states and use them as building blocks to construct more complicated ones.

3. Intelligent states with $\ell = 1/2$

Consider the simplest realization of $\hat{L}_x - \mathrm{i}\alpha\hat{L}_y$. Using basis states $|+\rangle$ and $|-\rangle$, for which

$$\hat{L}_z \mapsto \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \qquad \hat{L}_x \mapsto \frac{1}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \hat{L}_y \mapsto \frac{1}{2}\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix}, \tag{25}$$

we obtain the 2×2 matrix

$$\hat{L}_x - \mathrm{i}\alpha\hat{L}_y \mapsto \frac{1}{2}\begin{pmatrix}0 & 1-\alpha\\ 1+\alpha & 0\end{pmatrix}. \tag{26}$$

The (unnormalized) eigenstates, which are by definition intelligent states, are just

$$\begin{pmatrix}1\\ \frac{1+\alpha}{\sqrt{1-\alpha^2}}\end{pmatrix}, \qquad \begin{pmatrix}1\\ -\frac{1+\alpha}{\sqrt{1-\alpha^2}}\end{pmatrix}, \tag{27}$$

with respective eigenvalues

$$\lambda_+ = \lambda \equiv \tfrac{1}{2}\sqrt{1-\alpha^2}, \qquad \lambda_- = -\lambda. \tag{28}$$

Introducing the quantity

$$\mu = \frac{1+\alpha}{\sqrt{1-\alpha^2}}, \tag{29}$$

we obtain the normalized intelligent states as

$$\left|\psi_{-}^{1/2}(\mu)\right\rangle = \frac{1}{\sqrt{1+|\mu|^2}}\begin{pmatrix}1\\-\mu\end{pmatrix} = \frac{1}{\sqrt{1+|\mu|^2}}|+\rangle - \frac{\mu}{\sqrt{1+|\mu|^2}}|-\rangle, \tag{30}$$

$$\left|\psi_{+}^{1/2}(\mu)\right\rangle = \frac{1}{\sqrt{1+|\mu|^2}}\begin{pmatrix}1\\\mu\end{pmatrix} = \frac{1}{\sqrt{1+|\mu|^2}}|+\rangle + \frac{\mu}{\sqrt{1+|\mu|^2}}|-\rangle. \tag{31}$$

We note that, if $|\alpha| < 1$, μ is real and we can write

$$\left|\psi_{\pm}^{1/2}(\beta)\right\rangle = R_y(\pm\beta)|+\rangle = \mathrm{e}^{\mp \mathrm{i}\beta\hat{L}_y}|+\rangle, \tag{32}$$

with

$$\cos\tfrac{\beta}{2} = \frac{1}{\sqrt{1+|\mu|^2}}, \qquad \sin\tfrac{\beta}{2} = \frac{\mu}{\sqrt{1+|\mu|^2}}. \tag{33}$$

From equation (7), we see that the spin-$1/2$ *intelligent states are also coherent states when* μ *is real.*

On the other hand, when $|\alpha| \geqslant 1$, μ is purely imaginary and we have

$$\left|\psi_{\pm}^{1/2}(\beta)\right\rangle = R_x(\pm\beta)|+\rangle = \mathrm{e}^{\mp \mathrm{i}\beta\hat{L}_x}|+\rangle, \tag{34}$$

where, this time,

$$\cos\tfrac{\beta}{2} = \frac{1}{\sqrt{1+|\mu|^2}}, \qquad \mathrm{i}\sin\tfrac{\beta}{2} = \frac{\mu}{\sqrt{1+|\mu|^2}}. \tag{35}$$

From equation (7), we see that the spin-$1/2$ *intelligent states are also coherent states when* μ *is purely imaginary.*

4. General construction

4.1. Example: an intelligent states with $\ell = 5/2$

We can use the states $\left|\psi_{\pm}^{1/2}(\beta)\right\rangle$ of equations (30) and (31) to construct $\ell = 5/2$ intelligent states as follows. Consider the product

$$|\psi_{+++--}(\beta)\rangle = \left[\left|\psi_{+}^{1/2}(\beta)\right\rangle_1\left|\psi_{+}^{1/2}(\beta)\right\rangle_2\left|\psi_{+}^{1/2}(\beta)\right\rangle_3\right] \otimes \left[\left|\psi_{-}^{1/2}(\beta)\right\rangle_4\left|\psi_{-}^{1/2}(\beta)\right\rangle_5\right]. \tag{36}$$

Here, the index i in $\left|\psi_{+}^{1/2}(\beta)\right\rangle_i$ or $\left|\psi_{-}^{1/2}(\beta)\right\rangle_i$ labels one of five spin-$\frac{1}{2}$ subsystems. If we expand every $\left|\psi_{+}^{1/2}(\beta)\right\rangle_i$, every $\left|\psi_{-}^{1/2}(\beta)\right\rangle_i$ and distribute the product, the first term of the resulting expression is given by

$$\left|\ell = \tfrac{5}{2}, m = \tfrac{5}{2}\right\rangle = \cos^5\tfrac{\beta}{2}\left(|+\rangle_1|+\rangle_2|+\rangle_3|+\rangle_4|+\rangle_5\right). \tag{37}$$

This term is fully symmetric under permutation.

Let us use the shorthands

$$\hat{L}_{x,1} = \hat{L}_x \otimes \mathbb{1}_2 \otimes \mathbb{1}_3 \otimes \mathbb{1}_4 \otimes \mathbb{1}_5, \qquad \hat{L}_{x,2} = \mathbb{1}_1 \otimes \hat{L}_x \otimes \mathbb{1}_3 \otimes \mathbb{1}_4 \otimes \mathbb{1}_5 \tag{38}$$

etc, so that each $\hat{L}_{x,i}$ acts only on the ith subspace (of dimension 2). Let

$$\hat{L}_{x,\mathrm{A}} = \hat{L}_{x,1} + \hat{L}_{x,2} + \hat{L}_{x,3}, \qquad \hat{L}_{x,\mathrm{B}} = \hat{L}_{x,4} + \hat{L}_{x,5}, \tag{39}$$

and define

$$\hat{L}_x = \hat{L}_{x,\mathrm{A}} + \hat{L}_{x,\mathrm{B}}. \tag{40}$$

The collective operators $\hat{L}_y$ and $\hat{L}_z$ are defined similarly, as are $\hat{L}_{\pm}$:

$$\hat{L}_{\pm} = \hat{L}_x \pm \mathrm{i}\hat{L}_y. \tag{41}$$

Because the collective operators are fully symmetric under permutation of any two subspace index i in equation (40), and act on the symmetric state $\left|\frac{5}{2}, \frac{5}{2}\right\rangle$, every state of

angular momentum $\ell = 5/2$ will be symmetric under permutation. Thus, the order in which the $|+\rangle$'s or $|-\rangle$'s occur is unimportant.

4.1.1. The case $|\alpha| < 1$. With $|\alpha| < 1$, every $\left|\psi_{\pm}^{1/2}(\beta)\right\rangle$ is obtained by rotation about the y-axis. Thus, we can write

$$|\psi_{+++--}(\beta)\rangle = \left[R_y^{\mathrm{A}}(\beta)\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle_{\mathrm{A}}\right] \times \left[R_y^{\mathrm{B}}(-\beta)|1, 1\rangle_{\mathrm{B}}\right], \tag{42}$$

where we have directly coupled

$$\begin{aligned}[R_y(\beta)|+\rangle_1] \otimes [R_y(\beta)|+\rangle_2] \otimes [R_y(\beta)|+\rangle_3] &= R_y^{\mathrm{A}}(\beta)[|+\rangle_1|+\rangle_2|+\rangle_3], \\ &= R_y^{\mathrm{A}}(\beta)\left|\tfrac{3}{2}, \tfrac{3}{2}\right\rangle_{\mathrm{A}}, \end{aligned} \tag{43}$$

$$[R_y(-\beta)|+\rangle_4] \otimes [R_y(-\beta)|+\rangle_5] = R_y^{\mathrm{B}}(-\beta)|1, 1\rangle_{\mathrm{B}}. \tag{44}$$

Here, the rotation operator $R_y^{\mathrm{A}}(\beta) = \mathrm{e}^{-\mathrm{i}\beta\hat{L}_{y,\mathrm{A}}}$ while $R_y^{\mathrm{B}}(-\beta) = \mathrm{e}^{\mathrm{i}\beta\hat{L}_{y,\mathrm{B}}}$. Note that the states of equations (43) and (44) are both angular momentum coherent states.

Equation (42) can now be expanded as

$$\sum_{m_{\mathrm{A}}, m_{\mathrm{B}}} \left|\tfrac{3}{2}, m_{\mathrm{A}}\right\rangle_{\mathrm{A}}|1, m_{\mathrm{B}}\rangle_{\mathrm{B}}\, d^{3/2}_{m_{\mathrm{A}},3/2}(\beta) d^{1}_{m_{\mathrm{B}},1}(-\beta), \tag{45}$$

where

$$d^{\ell}_{m,m'}(\beta) \equiv \langle \ell, m|R_y(\beta)|\ell, m'\rangle \tag{46}$$

is the reduced Wigner d-function [19].

To project into the $\ell = 5/2$ subspace, we specialize the projector

$$\hat{\Pi}^{\ell} = \sum_{m=-\ell}^{\ell} |\ell, m\rangle\langle \ell, m| \tag{47}$$

to $\ell = 5/2$ so as to obtain

$$\left|\psi^{5/2}_{+++--}(\beta)\right\rangle \propto \sum_{m} |5/2, m\rangle \kappa^{5/2,m}_{3/2,1}(\beta), \tag{48}$$

where

$$\kappa^{\ell,m}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta) = \sum_{m_{\mathrm{A}}(m_{\mathrm{B}})} C^{\ell m}_{\ell_{\mathrm{A}},m_{\mathrm{A}};\ell_{\mathrm{B}},m_{\mathrm{B}}} \times d^{\ell_{\mathrm{A}}}_{m_{\mathrm{A}},\ell_{\mathrm{A}}}(-\beta) d^{\ell_{\mathrm{B}}}_{m_{\mathrm{B}},\ell_{\mathrm{B}}}(\beta), \tag{49}$$

and $C^{\ell m}_{\ell_{\mathrm{A}},m_{\mathrm{A}};\ell_{\mathrm{B}},m_{\mathrm{B}}}$ is an $su(2)$ Clebsch–Gordan coefficient.

A better, more compact notation for $\left|\psi^{5/2}_{+++--}\right\rangle$ is

$$\left|\psi^{5/2}_{+++--}\right\rangle \equiv \left|\psi^{5/2}_{3/2,1}(\beta)\right\rangle. \tag{50}$$

This emphasizes that only the total number of $|+\rangle_i$ states and the total number of $|-\rangle_j$ states are relevant for the construction of an intelligent state of angular momentum $\ell = \ell_{\mathrm{A}} + \ell_{\mathrm{B}}$. The state $\left|\psi^{5/2}_{++-+-}(\beta)\right\rangle$, for instance, can differ from $\left|\psi^{5/2}_{+++--}(\beta)\right\rangle$ by at most a phase.

To show that the state of equation (50) is intelligent, we note that the operator $\hat{\Pi}^{5/2}$ of equation (47) acts as the unit operator on any state completely in the $\ell = 5/2$ subspace, and annihilates any state with no part in this subspace. Hence, the collective operators $\hat{L}_y$ and $\hat{L}_x$ must commute with the projection $\hat{\Pi}^{5/2}$ of equation (47) since neither $\hat{L}_y$ nor $\hat{L}_x$ can change ℓ. Thus,

$$(\hat{L}_y - \mathrm{i}\alpha\hat{L}_x)\left|\psi^{5/2}_{3/2,1}(\beta)\right\rangle = \hat{\Pi}^{5/2}(\hat{L}_y - \mathrm{i}\alpha\hat{L}_x)|\psi_{3/2,1}(\beta)\rangle, \tag{51}$$

$$= (3\lambda_+ + 2\lambda_-)\left|\psi^{5/2}_{3/2,1}(\beta)\right\rangle. \tag{52}$$

The projection does not preserve the norm so $\left|\psi^{5/2}_{3/2,1}(\beta)\right\rangle$ must be normalized after the projection.

Since $R^{\mathrm{A}}_y(\beta)\left|\frac{3}{2},\frac{3}{2}\right\rangle_{\mathrm{A}}$ and $R^{\mathrm{B}}_y(-\beta)|1,1\rangle_{\mathrm{B}}$ are coherent, we see that $\left|\psi^{5/2}_{3/2,1}(\beta)\right\rangle$ is the result of coupling two *su*(2) coherent states.

4.1.2. The case $|\alpha| \geqslant 1$. In this case, we note that

$$\langle \ell, m|R_x(\beta)|\ell, \ell\rangle = \langle \ell, m|R_z(-\pi/2)R_y(\beta)R_z(\pi/2)|\ell, \ell\rangle, \tag{53}$$

$$= \mathrm{e}^{-\mathrm{i}\pi(\ell-m)/2}d^{\ell}_{m,\ell}(\beta), \tag{54}$$

so that, for instance,

$$\left|\psi^{5/2}_{3/2,1}(\beta)\right\rangle \propto \sum_m \left|\tfrac{5}{2}, m\right\rangle \mathrm{e}^{-\mathrm{i}\pi(\frac{5}{2}-m)/2}\kappa^{5/2,m}_{3/2,1}(\beta), \tag{55}$$

is intelligent by the same argument given for the $|\alpha| < 1$ case.

4.2. A general expression

More generally, it is now clear that if we start with $2\ell_{\mathrm{A}}$ copies of $\left|\psi^{1/2}_{+}(\beta)\right\rangle$ and $2\ell_{\mathrm{B}}$ copies of $\left|\psi^{1/2}_{-}(\beta)\right\rangle$, we can write assuming $|\alpha| < 1$ for simplicity)

$$\left[R^{\mathrm{A}}_y(\beta)|\ell_{\mathrm{A}}, \ell_{\mathrm{A}}\rangle\right] \otimes \left[R^{\mathrm{B}}_y(-\beta)|\ell_{\mathrm{B}}, \ell_{\mathrm{B}}\rangle\right], \tag{56}$$

and project into a good ℓ subspace using equation (47) to obtain an intelligent state of angular momentum $\ell = \ell_{\mathrm{A}} + \ell_{\mathrm{B}}$ as

$$\left|\psi^{\ell}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta)\right\rangle \propto \sum_m |\ell, m\rangle\kappa^{\ell,m}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta), \tag{57}$$

with $\kappa^{\ell,m}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta)$ given in equation (49).

Equations (56) and (57) show explicitly how *su*(2) intelligent states with angular momentum ℓ can be constructed by appropriately coupling *su*(2) coherent states. The state of equation (56) is explicitly intelligent and remains intelligent under projection by $\hat{\Pi}^{\ell}$ of equation (47), thus yielding equation (57).

We show in the appendix how $\kappa^{\ell,m}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta)$ can be reduced to

$$\kappa^{\ell,m}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta) = 2^{\ell}\frac{\sqrt{(2\ell_{\mathrm{B}})!\,(2\ell_{\mathrm{A}})!\,(\ell+m)!\,(\ell-m)!}}{(2\ell)!}d^{\ell}_{\ell_{\mathrm{B}}-\ell_{\mathrm{A}},m}\left(\frac{\pi}{2}\right)d^{\ell}_{m,\ell}(\beta). \tag{58}$$

Introducing the norm

$$\mathcal{N}^{\ell}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta) = \frac{1}{\sqrt{\sum_m \left|\kappa^{\ell,m}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta)\right|^2}}, \tag{59}$$

we obtain the final expression for our intelligent state as

$$\left|\psi^{\ell}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta)\right\rangle = \mathcal{N}^{\ell}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta)\sum_m |\ell, m\rangle\kappa^{\ell,m}_{\ell_{\mathrm{A}},\ell_{\mathrm{B}}}(\beta). \tag{60}$$

Finally, we note that the eigenvalue problem in the $\ell = \ell_{\mathrm{A}} + \ell_{\mathrm{B}}$ subspace has at most $2\ell + 1$ independent eigenvectors. Using equation (60), it is clear that (except when $\beta = 0$ or π) we can construct exactly the right number of linearly independent states by selecting in turn $(\ell_{\mathrm{A}}, \ell_{\mathrm{B}})$ to be $(\ell, 0), (\ell - 1/2, 1/2), \ldots, (0, \ell)$. Hence, all $2\ell + 1$ intelligent states are coupled *su*(2) coherent states.

When $\beta = 0$ or π, α is ∓ 1, the operator $\hat{L}_x - \mathrm{i}\alpha\hat{L}_y$ is nilpotent and has a single eigenvector: either $|\ell, \ell\rangle$ or $|\ell, -\ell\rangle$. This is (indirectly) illustrated in figures 1 and 2, where it is that all uncertainty curves merge to a single curve at $\beta = \pi$ (or $\alpha = 1$).

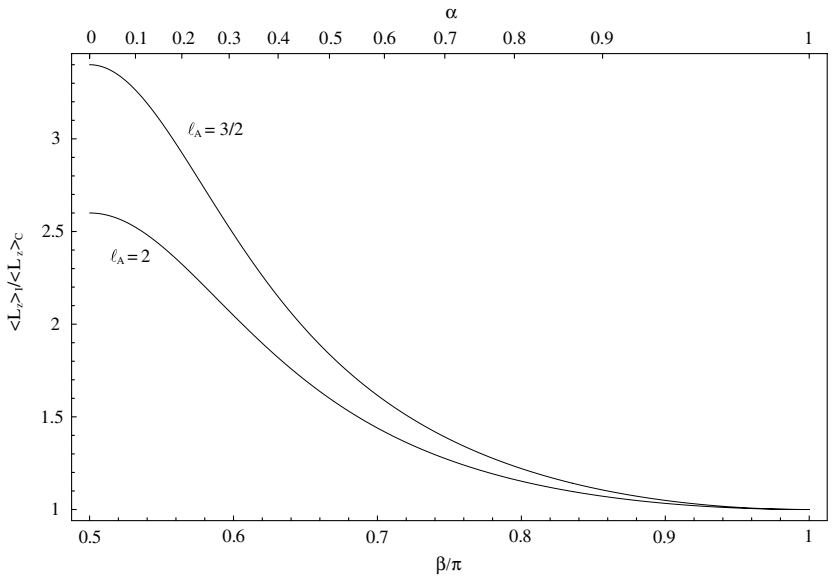

Figure 1. The ratio $|\langle\hat{L}_z\rangle|_I/|\langle\hat{L}_z\rangle|_c$ as a function of β/π or α for $\ell = 5/2$ and various values of ℓ_A and ℓ_B so that $\ell_A + \ell_B = 5/2$.

5. Selected results

5.1. Expectations and standard deviations

The intelligent state of equation (60) is an eigenstate of $\hat{L}_x - i\alpha\hat{L}_y$ with eigenvalue

$$\lambda_{\ell_A,\ell_B} = \lambda(2\ell_A - 2\ell_B). \tag{61}$$

If we assume $|\alpha| \leqslant 1$, then λ is real. Combining equations (28), (29) and (33), we obtain

$$\lambda_{\ell_A,\ell_B} = (\ell_A - \ell_B)\sin\beta. \tag{62}$$

Since α, $\langle\hat{L}_x\rangle$ and $\langle\hat{L}_y\rangle$ are real, this can be compared with $\lambda_{\ell_A,\ell_B} = \langle\hat{L}_x\rangle - i\alpha\langle\hat{L}_y\rangle$ to give

$$\langle\hat{L}_x\rangle = \tfrac{1}{2}(\ell_B - \ell_A)\sin\beta, \qquad \langle\hat{L}_y\rangle = 0. \tag{63}$$

If, on the other hand, $|\alpha| \geqslant 1$, we have

$$\langle\hat{L}_y\rangle = -\tfrac{1}{2}(\ell_B - \ell_A)\sin\beta, \qquad \langle\hat{L}_x\rangle = 0. \tag{64}$$

Furthermore, using equations (14) and (15), one finds that the intelligent states generally satisfy

$$(\Delta L_y)^2 = -\frac{1}{2\alpha}\langle\hat{L}_z\rangle, \qquad (\Delta L_x)^2 = -\frac{1}{2}\alpha\langle\hat{L}_z\rangle. \tag{65}$$

This allows computation of all pertinent quantities from $\langle\hat{L}_z\rangle$, which is simply given by

$$\langle\hat{L}_z\rangle = \left(\mathcal{N}^{\ell}_{\ell_A\ell_B}(\beta)\right)^2 \left(\sum_m m\left|\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)\right|^2\right). \tag{66}$$

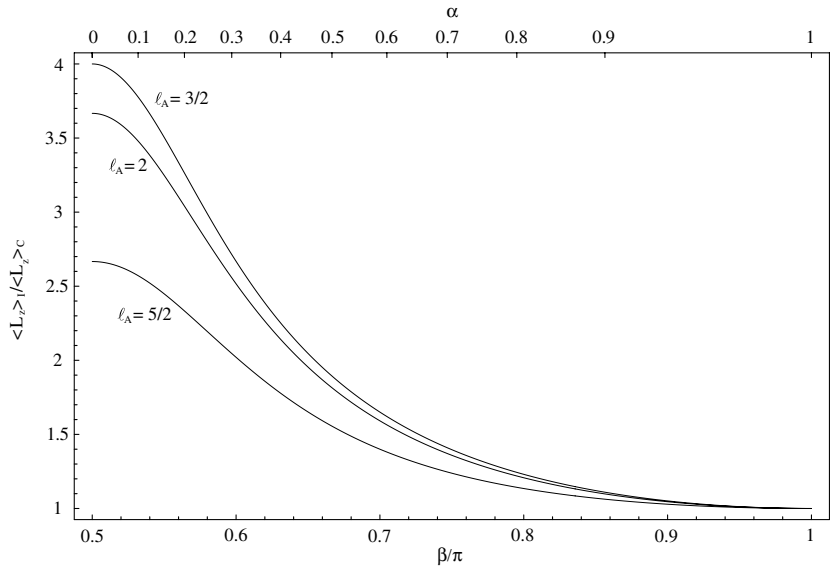

Figure 2. The ratio $|\langle\hat{L}_z\rangle|_{\mathrm{I}}/|\langle\hat{L}_z\rangle|_{\mathrm{c}}$ as a function of β/π or α for $\ell = 3$ and various values of ℓ_{A} and ℓ_{B} so that $\ell_{\mathrm{A}} + \ell_{\mathrm{B}} = 3$.

5.2. Numerical results

Figures 1 and 2 illustrate typical results. The figures give the ratio of the uncertainty products $(\Delta L_x \Delta L_y)_{\mathrm{I}}$ of intelligent states to the coherent state $(\Delta L_x \Delta L_y)_{\mathrm{c}}$, for which $\ell_{\mathrm{A}} = \ell$. These ratios are just the ratios of $\langle\hat{L}_z\rangle$. For the coherent state, one rapidly finds

$$\langle\hat{L}_z\rangle_{\mathrm{c}} = \frac{\ell}{2}\cos\beta, \tag{67}$$

for $|\alpha| < 1$.

In figure 1, the ratios for intelligent states of angular momentum $\ell = 5/2$ with $(\ell_{\mathrm{A}} = 2, \ell_{\mathrm{B}} = 1/2)$ and $(\ell_{\mathrm{A}} = 3/2, \ell_{\mathrm{B}} = 1)$ are given. The results are unchanged if one switches ℓ_{A} and ℓ_{B}. The curves $\alpha < 0$ are identical to those for $\alpha > 0$. Furthermore, the results with $|\alpha| > 1$ can be obtained from those with $|\alpha| < 1$ by the transformation $\alpha \to -1/\alpha$, so the range $0 \leqslant \alpha \leqslant 1$ captures all qualitative features of the curves. Figure 2 is similar to 1, except that $\ell = 3$. The symmetries of figure 1 are also present in figure 2.

One immediately observes that the uncertainty products for intelligent states (with $\ell_{\mathrm{A}} \neq \ell$) are always greater than the corresponding product for the coherent state (with $\ell_{\mathrm{A}} = \ell$). Insofar as the product $\Delta L_x \Delta L_y$ goes, the 'worst' intelligent state is the state for which ℓ_{A} and ℓ_{B} are as close as possible. We have not been able to prove this analytically because the expression (66) for $\langle\hat{L}_z\rangle$ is difficult to manipulate. However, we have verified that this observation holds over a wide range of values of ℓ. Other curves illustrating this behaviour can be found in [10].

It is not difficult to show that the maximum of the product $\Delta L_x \Delta L_y$ is simply $\frac{1}{2}\ell$. Indeed, by equation (3), it is clear that the product is maximal when $|\langle\hat{L}_z\rangle|$ is maximal. This maximum is reached for the states $|\ell, \pm\ell\rangle$. From equations (56) and (57), it immediately follows that this will occur when $\beta = 0$ or $\beta = \pi$. This, implies by equation (33) that $\mu = 0$ or $\mu = \infty$ which in turn,by equation (29), implies $\alpha = \mp 1$.

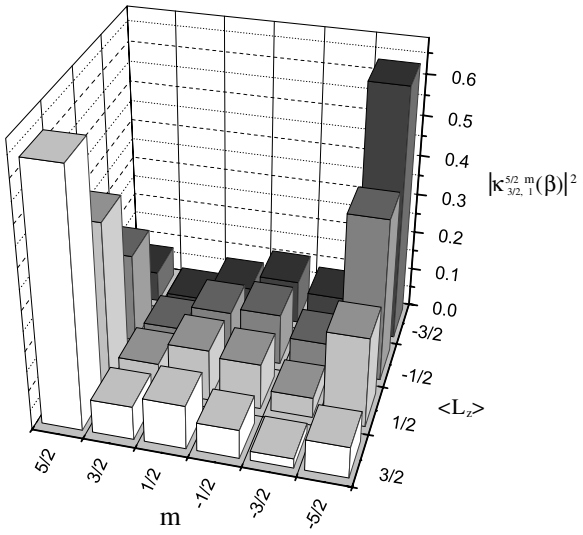

Figure 3. The populations of m substates $|\kappa^{5/2m}_{3/2,1}(\beta)|^2$ for different values of m and $\ell = 5/2$. The values of β were selected so that $\langle \hat{L}_z \rangle = \pm 3/2, \pm 1/2$.

As $\alpha \to \pm 1$, all intelligent states converge to a single state. When $\alpha = \pm 1$ precisely, the operator $\hat{L}_x - \mathrm{i}\alpha \hat{L}_y$ becomes the nilpotent $\hat{L}_+$ or $\hat{L}_-$ respectively, both of which have only one non-zero eigenvector.

Figure 3 shows the population of various m substates in the intelligent state $\left|\psi^{5/2}_{3/2,1}(\beta)\right\rangle$. For clarity, we have restricted the calculations to angles β chosen so that $\langle \hat{L}_z \rangle = \pm 3/2, \pm 1/2$.

This figure illustrates a very general symmetry: $\left|\kappa^{\ell,m}_{\ell_\mathrm{A},\ell_\mathrm{B}}(\beta)\right|^2 = \left|\kappa^{\ell,-m}_{\ell_\mathrm{A},\ell_\mathrm{B}}(-\beta)\right|^2$. This can be traced back to symmetries of the d-functions entering in the construction of the $\kappa^{\ell,m}_{\ell_\mathrm{A},\ell_\mathrm{B}}(\beta)$ coefficients.

6. Discussion and conclusion

Let us construct the operators

$$\hat{K}_i = \hat{L}_{i\mathrm{A}} - \hat{L}_{i\mathrm{B}}. \tag{68}$$

The operators $\{\hat{K}_x, \hat{K}_y, \hat{K}_z\}$ do not close under commutation. However, $\hat{K}_x, \hat{K}_y, \hat{L}_z$ do close on an angular momentum algebra, which we call the K-angular momentum $su(2)_K$. This set is interesting because our intelligent states are constructed using a K-rotation about y. Indeed, defining $\hat{K}_\pm$ in the usual manner, one can see that the state

$$|\ell_\mathrm{A}, \ell_\mathrm{A}\rangle |\ell_\mathrm{B}, \ell_\mathrm{B}\rangle, \tag{69}$$

is an eigenstate of $\hat{L}_z$ with eigenvalue $m_K = \ell_\mathrm{A} + \ell_\mathrm{B} = \ell$. Because (69) is killed by $\hat{K}_+$, it can be identified with the state $|\ell, m_K = \ell\rangle_K$ of K-angular momentum. In particular, our starting state

$$\begin{aligned}\left[R^\mathrm{A}_y(\beta)\,|\ell_\mathrm{A}, \ell_\mathrm{A}\rangle\right]\left[R^\mathrm{B}_y(-\beta)\,|\ell_\mathrm{B}, \ell_\mathrm{B}\rangle\right] &= \exp[-\mathrm{i}\beta(\hat{L}_{y\mathrm{A}} - \hat{L}_{y\mathrm{B}})]|\ell, \ell\rangle_K \\ &= \exp[-\mathrm{i}\beta \hat{K}_y]|\ell, \ell\rangle_K \end{aligned} \tag{70}$$

and is thus an $su(2)_K$ coherent state.

Unfortunately, the K-angular momenta do not commute with the collective angular momenta $\hat{L}_i$. Although (69) is simultaneously a state with 'good' total ℓ, m_ℓ and 'good' $\ell_K = \ell, m_K$, other $|\ell, m_K \neq \ell\rangle_K$ states generated by the action of $\hat{K}_-$ do not have 'good' ℓ, m_ℓ; hence the need for the projection into the subspace of good ℓ-angular momentum.

In [10], a method of constructing all $su(2)$ intelligent states of angular momentum ℓ was proposed. The basic polynomials ξ^x and η^y are related to the direct product of x copies of $|+\rangle$ and the direct product of y copies of $|-\rangle$, respectively, via the correspondences

$$|+\rangle \leftrightarrow \xi, \qquad |-\rangle \leftrightarrow \eta, \qquad |\ell, m\rangle \leftrightarrow \frac{\xi^{\ell+m}\eta^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}}. \tag{71}$$

Using this, we can write, for $|\alpha| < 1$, the intelligent state $\left|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\right\rangle$ as the product

$$\left|\psi^{\ell}_{\ell_A,\ell_B}(\beta)\right\rangle = \left(\xi\cos\tfrac{\beta}{2} + \eta\sin\tfrac{\beta}{2}\right)^{2\ell_A}\left(\xi\cos\tfrac{\beta}{2} - \eta\sin\tfrac{\beta}{2}\right)^{2\ell_B}. \tag{72}$$

There is no need for projection as the result is a polynomial of total degree $2\ell = 2(\ell_A + \ell_B)$. It is well known that the polynomials of the form $\xi^x\eta^y$, with $x + y = 2\ell$, span a basis for an $su(2)$ representation of angular momentum ℓ. The combinatorics involved in the expansion of equation (72) and the conversion of various $\xi^x\eta^y$ to angular momentum states yield precisely equation (60). Thus, we recover in a much more transparent way the construction and calculations of [10]. (An expression similar to equation (72) can easily be found for $|\alpha| \geqslant 1$.)

The simple form of equations (60), (65) and (66) illustrate the economy inherent to an approach based on coupling. These results can be contrasted, for instance, with the corresponding expressions of [8] or the application done by [22] of $su(2)$ intelligent states in nuclear physics.

Our results, which only require a table of Clebsch–Gordan coefficients and expressions for Wigner D-functions, represent the simplest example of what could be a systematic algorithm for the construction of intelligent states of observables that are elements of other Lie algebras [20] or even deformed algebras [21]. In other words, the procedure presented here is easily generalizable. Indeed, using the results of [23, 24], the properties of some $SU(3)$ intelligent states will be the topic of a forthcoming paper [20].

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Appendix. The coefficient $\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta)$

The expression of equation (58) can be manipulated into a more transparent form using [19]

$$d^{\ell_B}_{m_B,\ell_B}(-\beta) = (-1)^{m_B-\ell_B} d^{\ell_B}_{m_B,\ell_B}(\beta), \tag{A.1}$$

$$d^{\ell_A}_{m_A,\ell_A}(\beta) d^{\ell_B}_{m_B,\ell_B}(\beta) = C^{\ell m}_{\ell_A,m_A;\ell_B,m_B} \times d^{\ell}_{m,\ell}(\beta), \tag{A.2}$$

where $\ell = \ell_A + \ell_B$ and $C^{\ell\ell}_{\ell_A,\ell_A;\ell_B,\ell_B} = 1$ have been used. Thus,

$$\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta) = d^{\ell}_{m,\ell}(\beta) \times \left[\sum_{m_A(m_B)} (-1)^{m_B-\ell_B}\left(C^{\ell m}_{\ell_A,m_A;\ell_B,m_B}\right)^2\right]. \tag{A.3}$$

A little more progress can be made because Clebsch–Gordan coefficients for which $\ell = \ell_A + \ell_B$ have known expressions [19]. Using this and the condition $m = m_A + m_B$, we obtain

$$\kappa^{\ell,m}_{\ell_A,\ell_B}(\beta) = \frac{d^{\ell}_{m,\ell}(\beta)}{\binom{2\ell}{\ell-m}} \left[\sum_{n=0}^{2\ell_B} (-1)^n \binom{2\ell_A}{\ell-m-n} \binom{2\ell_B}{n} \right]. \tag{A.4}$$

The coefficient in the bracket can be identified with the coefficient of $x^{\ell-m}$ in the expansion of $(1+x)^{2\ell_A}(1-x)^{2\ell_B}$. In particular, when $\ell_A = \ell_B$, ℓ is integer and there can be no odd powers of x, so that no odd values of m will appear in the expansion.

Finally [19],

$$\sum_{n=0}^{2\ell_B} (-1)^n \binom{2\ell_A}{\ell-m-n} \binom{2\ell_B}{n} = 2^{\ell} \sqrt{\frac{(2\ell_B)!\,(2\ell_A)!}{(\ell+m)!\,(\ell-m)!}} d^{\ell}_{\ell_B-\ell_A,m}\left(\frac{\pi}{2}\right). \tag{A.5}$$

Inserting this into $\kappa^{\ell,m}_{\ell_A\ell_B}(\beta)$ gives equation (58).

Note that the appearance of a rotation by $\pi/2$ about the $\hat{y}$ axis:

$$\begin{aligned} d^{\ell}_{m,\ell_B-\ell_A}\left(\frac{\pi}{2}\right) &= d^{\ell}_{\ell_B-\ell_A,m}\left(-\frac{\pi}{2}\right) \\ &= \langle \ell, \ell_B - \ell_A | e^{-i\frac{\pi}{2}\hat{L}_y} | \ell, m \rangle, \end{aligned} \tag{A.6}$$

is reminiscent of an expression found in [8].

Lastly, although we have limited ourselves to expressions where $\ell = \ell_A + \ell_B$, the factor $\ell_B - \ell_A$ makes it clear that, up to a normalization, it is only the difference between angular momenta that is here relevant. More precisely, if one considers $\ell'_A = \ell_A + j$, $\ell'_B = \ell_B + j$, then the tensor product $\ell'_A \otimes \ell'_B$ will contain a subspace of angular momentum ℓ. The coupled states in this subspace are also intelligent, but are simply proportional to the state obtained by coupling $\ell_A \otimes \ell_B$. In other words, no new state is found by considering cases other than $\ell = \ell_A + \ell_B$.

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